Mellin Transform

Definition:

$$\mathcal{M}_{\{f\}}(s) = \int_0^\infty dx\, x^{s-1} f(x) \qquad = \varphi(s)$$

Imposing convergence of this integral results in the definition of a "fundamental strip" for which the Mellin transform exists for any complex number. Often $\varphi(s)$ can be extended to the rest of the complex plane.

The transform is related to the Fourier and bilateral Laplace transforms.

Fourier: Take $s = \sigma + it \qquad x = e^{-y}$

$$\mathcal{M}_{\{f\}}(s) = \int_{-\infty}^{+\infty} f(e^{-y})\, e^{-\sigma y}\, e^{-ity}\, dy$$

$$= \mathcal{F}_{\{(f\circ e)\cdot e\}}(\sigma, t)$$

In particular, from the fourier inversion formula, one can write the inverse mellin transform as:

$$\mathcal{M}^{-1}_{\{\varphi\}}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, \varphi(s)\, x^{-s}$$

where c is a real number belonging to the fundamental strip.

The mellin transform is of great importance in number theory to study the behavior of harmonic sums. Take for instance:

$$F(x) = \sum_k \lambda_k f(\mu_k x)$$

The mellin transform gives:

$$\mathcal{M}_{\{F\}}(s) = \mathcal{M}_{\{f\}}(s) \sum_k \frac{\lambda_k}{\mu_k^s}$$

which is offe connect to the stud of series.

Mellin-Barnes Integrals

The integration procedure we develop is a spin-off of an intuition due to Gauss for writing functions in integral forms over the complex plane.

~~Theog~~ From the general Cauchy integral formula, it is known that:

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \varphi(s)\, ds = \sum_{k} \underset{s=a_k}{\mathrm{Res}}[\varphi(s)]$$

where $\{a_k\}$ are the simple poles of $\varphi(s)$ belonging to the inside of the prescribed contour $\mathcal{C}$

Take then: for simplicity:

$$\varphi(s) = \Gamma(s)\, f(s)$$

where $f(s)$ is analytical inside the contour $\mathcal{C}$

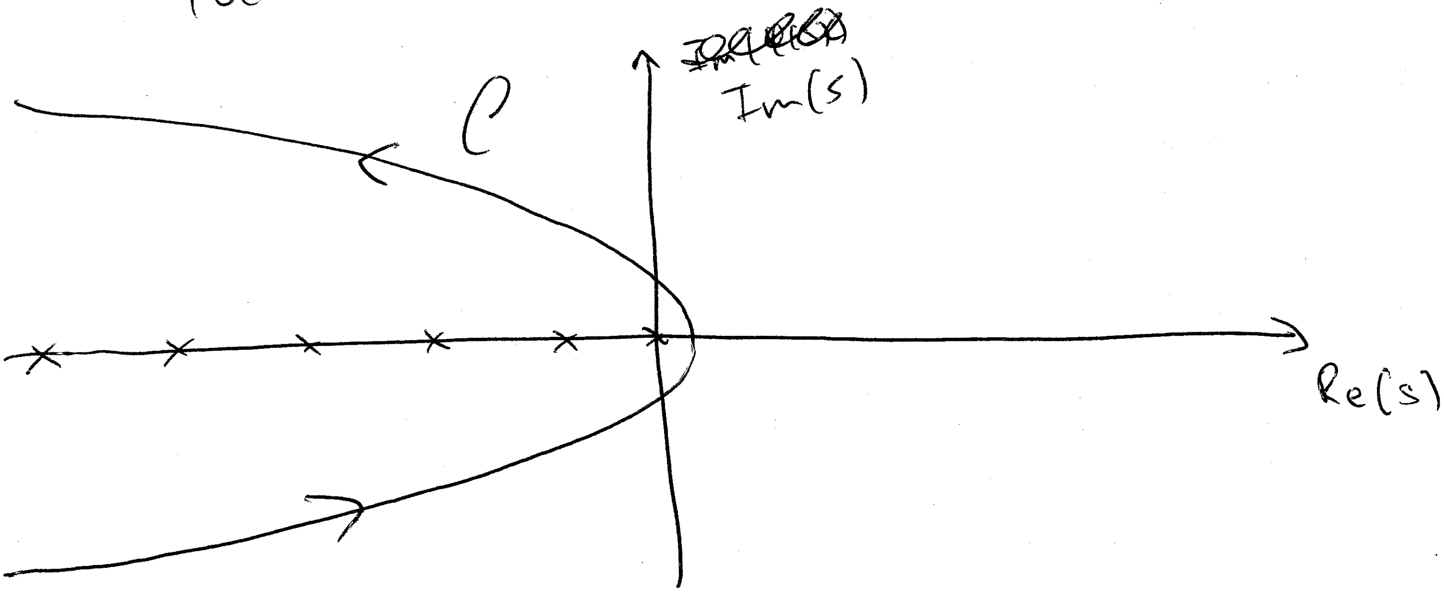
Then we have that:

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \Gamma(s)\, f(s)\, ds = \sum_{k \in \mathbb{K}} \frac{(-1)^k}{k!} f(-k)$$

Where $\mathbb{K} = \{ l \in \mathbb{Z} \text{ inside } \mathcal{C} \}$

By adapting the contour, we have the ability to make the sum infinite.

Take $\mathcal{C}$ to be of the form



where now $f(s)$ is analytic over the whole complex plane.

Then:

$$\frac{1}{2\pi i}\oint_C \Gamma(s)\, f(s)\, ds = \sum_{k=0}^{\infty} \frac{(-)^k}{k!} f(-k).$$

It is apparent now that choosing appropriate ~~forms~~ polynomial forms for $f(-k)$ we can relate the contour integral to the taylor expansion of a great number of functions

Ex:

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-)^k}{k!} x^k$$

choosing then $f(s) = (x^{-1})^s$ we have:

$$e^{-x} = \frac{1}{2\pi i}\oint_C \Gamma(s)\, x^{-s}\, ds$$

Ex:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{k!}{(2k+1)!} x^{2k}$$

~~Hence~~ placing $f(s) = \frac{\Gamma(1-s)}{\Gamma(2(1-s))} x^{-2s}$

$$\sin(x) = \frac{x}{2\pi i} \oint_C \Gamma(s) \frac{\Gamma(1-s)}{\Gamma(2(1-s))} x^{-2s} \, ds$$

~~The name of~~ These integral transforms are generally called Mellin-Barnes integrals.

The main algorithm I want to talk about ~~relies~~ revolves on viewing such integrals as inverse mellin transforms of some function.

Let us then consider a converging integral of the form:

$$I_\alpha = \int_0^\infty dx \, x^{\alpha-1} f(x)$$

$f(x)$ can be written in Mellin-Barnes form such that we have:

$$I_\alpha = \int_0^\infty dx\, x^{\alpha-1} \frac{1}{2\pi i} \oint_{\mathcal{C}} \varphi(s)\, x^{-s}\, ds$$

Since the contour of our M-B integral can be deformed into that of the inverse mellin transform, our required integral is simply the mellin transform of an inverse mellin transform!

$$I_\alpha = \mathcal{M}_{\{\mathcal{M}^{-1}_{\{\varphi\}}\}}(\alpha)$$

So our integral simply reduces to the form:

$$\boxed{I_\alpha = \varphi(\alpha)}$$

Lets see this in action:

$$\int_0^\infty e^{-x}\,dx = \lim_{\alpha\to 1}\int_0^\infty dx\, x^{\alpha-1}\frac{1}{2\pi i}\oint_{\mathcal{C}} \Gamma(s)\, x^{-s}\, ds$$

$$= \lim_{\alpha\to 1}\Gamma(\alpha) = 1$$

$$\int_0^\infty dx\,\frac{\sin x}{x} = \lim_{\alpha\to 0}\int_0^\infty dx\, x^{\alpha-1}\frac{1}{2\pi i}\oint_{\mathcal{C}} \Gamma(s)\,\frac{\Gamma(1-s)}{\Gamma(2(1-s))}\, x^{1-2s}\, ds$$

placing

$$1-2s = -\eta \;\rightarrow\; ds = \frac{1}{2}\,d\eta\,,\quad s = \frac{1+\eta}{2}$$

$$\int_0^\infty dx\,\frac{\sin x}{x} = \lim_{\alpha\to 0}\int_0^\infty dx\, x^{\alpha-1}\left(\frac{1}{2}\right)\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\Gamma\left(\frac{1+\eta}{2}\right)\Gamma\left(\frac{1-\eta}{2}\right)}{\Gamma(1-\eta)}\, x^{-\eta}\, d$$

$$= \lim_{\alpha\to 0}\frac{\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)}{2\,\Gamma(1-\alpha)} = \frac{\pi}{2}$$

A tougher integral:

$$\int_0^\infty dx\, x^{\alpha-1} J_\beta(x)$$ where $J_\beta(x)$ is a bessel function of the first kind.

Its series definition is:

$$J_\beta(x) = \left(\frac{x}{2}\right)^\beta \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \frac{1}{\Gamma(k+\beta+1)} \left(\frac{x}{2}\right)^{2k}$$

$$= \left(\frac{x}{2}\right)^\beta \frac{1}{2\pi i} \oint_C ds\, \frac{\Gamma(s)}{\Gamma(\beta-s+1)} \left(\frac{x}{2}\right)^{-2s}$$

Changing variables:

$$2s = \eta \rightarrow \begin{cases} ds = \frac{1}{2} d\eta \\ s = \frac{\eta}{2} \end{cases}$$

$$J_\beta(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\beta \frac{1}{2\pi i} \oint_C d\eta\, \frac{\Gamma(\eta/2)}{\Gamma(\beta-\frac{\eta}{2}+1)} 2^\eta x^{-\eta}$$

Hence:

$$\int_0^\infty dx\, x^{\alpha-1} J_\beta(x) = 2^{-(\beta+1)} \int_0^\infty dx\, x^{\alpha+\beta-1} \frac{1}{2\pi i} \oint_C \frac{\Gamma(\eta/2)}{\Gamma(\beta-\frac{\eta}{2}+1)} 2^\eta x^{-\eta}$$

$$= \boxed{2^{\alpha-1} \frac{\Gamma\left(\frac{\alpha+\beta}{2}\right)}{\Gamma\left(\frac{\beta-\alpha}{2}+1\right)}}$$

Integrals of products

Consider a more complex integral of the form:

$$I_\alpha = \int_0^\infty d\dot{x}\ x^{\alpha-1}\ f(x)\, g(x)$$

where we ~~expend~~ expand f, g in their M-B integral forms:

$$f(x) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \varphi(s)\, x^{-s}\, ds$$

$$g(x) = \frac{1}{2\pi i}\oint_{\mathcal{C}'} \psi(\eta)\, x^{-\eta} d\eta$$

Plugging in and pulling out one of the M-B integrals:

$$I_\alpha = \frac{1}{2\pi i}\oint_{\mathcal{C}'} d\eta\, \psi(\eta) \times \left\{ \int_0^\infty x^{\alpha-\eta-1} dx \frac{1}{2\pi i}\oint_{\mathcal{C}} \varphi(s)\, x^{-s}\, ds \right\}$$

Applying the usual tecnique within the parentheses:

$$I_\alpha = \frac{1}{2\pi i} \oint_{\mathcal{C}'} d\eta \, \Psi(\eta)\, \varphi(\alpha-\eta)$$

Since the ~~mellin~~ M-B integral forms are always expressed as ratios of products of gamma functions, the product $\Psi(\eta)\varphi(\alpha-\eta)$ will also be of such a form.

As such, I_α is simply the Mellin-Barnes integral form of some function calculated @ $x=1$.

Ratios of ~~gamma~~ products of gamma functions can often be simplified and reduced to commonly known functions (in their M-B form).

In a worst case scenario, from the definition of the hypergeometric functions, it can be easily seen how M-B forms can always be written as hypergeometric functions. As ~~sit~~ such, the method here described ~~extends~~ grants the user the possibility of writing any function and any integral solution in hypergeometric function form.

Programs like MATHEMATICA take explicit advantage of this property of M-B integral forms to manipulate and simplify equations.